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(32) 4.2 The Rolle's Theorem (A)

Roll's Theorem For $f(x)$	If (1) f is continuous on $[a, b]$ (2) f is differentiable on (a, b) (3) $f(a) = f(b)$ Then $\exists\, c \in (a, b)$ Such That $f'(c) = 0$

Roll's Theorem For $f'(x)$	If (1) f' is continuous on $[a, b]$ (2) f' is differentiable on (a, b) (3) $f'(a) = f'(b)$ Then $\exists\, c \in (a, b)$ Such That $f''(c) = 0$

Roll's Theorem For $f''(x)$	If (1) f'' is continuous on $[a, b]$ (2) f'' is differentiable on (a, b) (3) $f''(a) = f''(b)$ Then $\exists\, c \in (a, b)$ Such That $f'''(c) = 0$

Example 1	State the Rolle's Theorem

Solution

If
(1) f is continuous on $[a, b]$
(2) f is differentiable on (a, b)
(3) $f(a) = f(b)$
Then
$\exists\, c \in (a, b)$ Such That $f'(c) = 0$

Example 2 40 August 7, 2011	(4 Points) Let $f(x) = \tan(x)$. (a) Determine which conditions of Roll's Theorem are satisfied on the interval $[\,0\,,\pi\,]$. (b) Show that there is no $c \in (\,0\,,\pi\,)$ such that $f'(c) = 0$? Does this contradict Roll's Theorem ? (Justify your answer)

Solution

(a) conditions of Roll's Theorem

(1) f discont. at $x = \frac{\pi}{2}$

(2) f not diff. at $x = \frac{\pi}{2}$

(3) $f(0) = f(\pi) = 0$

(b)

$f(x) = \tan(x)$

$f^{\backslash}(x) = \sec^2 x \neq 0 \quad \forall\, x \in \mathcal{R}$

this does not contradict Roll's Theorem since the hypotheses of Roll's Theorem are not satisfied

Example 3 20 January 3, 2001	Determine whether $f(x) = x^2 + 4\sqrt{5 - x^2}$ satisfies the hypotheses of Rolle's theorem on $[-2\,,2\,]$ and, if so , Find the numbers c satisfying the conclusion of the Theorem

Solution

$f(x) = x^2 + 4\sqrt{5 - x^2} \qquad D_f \ [-\sqrt{5}, \sqrt{5}]$

$f^{\backslash}(x) = 2x + \dfrac{4(-2x)}{2\sqrt{5 - x^2}} = 2x - \dfrac{4}{\sqrt{5 - x^2}}$

$f(-2) = 4 + 4\sqrt{5 - 4} = 8$

$f(2) = 4 + 4\sqrt{5 - 4} = 8$

(1) f cont. on $[-2{,}2]$

(2) f diff. on $(-2{,}2)$

(3) $f(-2) = f(2)$

$\therefore f$ satisfies all the hypotheses of Rolle's theorem on $[-2\,,2\,]$

$\therefore\ \exists\ c\ \in\ (-2{,}2)$ such that $f^{\backslash}(c) = 0$

$2c - \dfrac{4}{\sqrt{5 - c^2}} = 0$

$2c = \dfrac{4}{\sqrt{5 - c^2}}$

$4c^2 = \dfrac{16}{5 - c^2}$

$20c^2 - 4c^4 = 16$

$4c^4 - 20c^2 + 16 = 0$

$c^4 - 5c^2 + 4 = 0$

$(c^2 - 4)(c^2 - 1) = 0$

$\therefore c = \pm 1 \ \in\ (-2{,}2)$

Example 4 19 May 13, 1999 28 Dec 20, 2001	if $f(x) = \cos 2x + 2\cos x$, Show that f satisfies the conditions of Rolle's theorem on interval $[0, 2\pi]$ and find a number $c \in (0, 2\pi)$ that satisfies the conclusion of the theorem

Solution

$f(x) = \cos 2x + 2\cos x$
$f^{\backslash}(x) = -2\sin 2x - 2\sin x$
$f(0) = \cos(0) + 2\cos(0) = 1 + 2 = 3$
$f(2\pi) = \cos(4\pi) + 2\cos(2\pi) = 1 + 2 = 3$

(1) f cont. on $[0,2\pi]$
(2) f diff. on $(0,2\pi)$
(3) $f(0) = f(2\pi)$
$\therefore \exists\ c \in (0,2\pi)$ such that $f^{\backslash}(c) = 0$
$-2\sin 2c - 2\sin c = 0$
$\sin 2c + \sin c = 0$
$2\sin c\cos c + \sin c = 0$
$\sin c\,(2\cos c + 1) = 0$
$\sin c = 0 \quad , \quad \cos c = \frac{-1}{2}$
$c = \pi \quad , \quad \pi - \frac{\pi}{3} \quad , \quad \pi + \frac{\pi}{3}$
$\therefore c = \pi \ , \frac{2\pi}{3} \quad , \quad \frac{4\pi}{3}$

Example 5 41 July 19, 2007	Let $f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 2 - x & \text{if } 1 < x \le 2 \end{cases}$ (a) Show that $f(2) = f(0)$ (b) Show that $f^{\backslash}(c) \neq 0$ for all $c \in (0, 2)$ (C) Does this not contradict the Roll's theorem ? Explain

Solution

(a) $f(0) = 0 \quad , \quad f(2) = 2 - 2 = 0 \qquad \therefore f(2) = f(0)$
(b) $f^{\backslash}(x) = \begin{cases} 1, & 0 < x < 1 \\ -1, & 1 < x < 2 \end{cases}$
$f^{\backslash}(1) \quad D.N.E$
f not diff. at $x = 1 \rightarrow f$ not diff. on $(0 , 2)$

$f^{\backslash}(C) \neq 0$
this does not contradict Roll’s Theorem since the hypotheses of Roll’s Theorem are not satisfied

Example 6 32 December 18, 2003	Let $f(x) = 3x + \frac{9}{2}(x-1)^{\frac{5}{3}} + 1$ (a) Show that there is no $c \in (0, 2)$ such that $f''(c) = 0$ (b) Explain why this result does not contradict the Rolle's theorem

Solution

$f(x) = 3x + \frac{9}{2}(x-1)^{\frac{5}{3}} + 1$

$f'(x) = 3 + \frac{15}{2}(x-1)^{\frac{2}{3}}$

$f''(c) = \frac{15}{2} \cdot \frac{2}{3}(x-1)^{-\frac{1}{3}} = \frac{5}{(x-1)^{\frac{1}{3}}}$

$\therefore \; f''(x) \neq 0$

f' not diff. $at\ x = 1$

$f'(0) = 3 + \frac{15}{2}(-1)^{2/3} = 3 + \frac{15}{2}$

$f'(2) = 3 + \frac{15}{2}(1)^{2/3} = 3 + \frac{15}{2}$

f' cont. on $[0,2]$

f' not diff. on $(0, 2)$

$f'(0) = f'(2)$

this does not contradict Roll's Theorem since the hypotheses of Roll's Theorem are not satisfied

Homework

1
11 May 15 , 1997
Does The Rolle's Theorem apply to $f(x)=\sqrt{9-x^2}$ on $[-3,3]$?
If yes , find a number c which satisfies the conclusion of the theorem
And if not explain why not .

2
9 January 8, 1994
Let $f(x)=x^2+2\sqrt{9-x^2}$ $-3\le x\le 3$, Find the numbers $c\in(-3\,,\,3)$ satisfying the conclusion of the Rolle's theorem for f

3
7 July 25, 1996
Let $f(x)=\cos\frac{x}{2}$, find all real numbers $c\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
that satisfy the conditions of Rolle's theorem for f

4
19 May 13, 1999 & 28 Dec 20, 2001
if $f(x)=\cos 2x+2\cos x$, Show that f satisfies the conditions of Rolle's theorem on interval $[0\,,2\pi]$ and find a number $c\in(0\,,2\pi)$ that satisfies the conclusion of the theorem

5
8 July 30 1996
Let $f(x)=\sin\frac{x}{4}+\cos\frac{x}{4}$, find all real numbers $c\in(0\,,\,2\pi)$
that satisfy the conditions of Rolle's theorem for f

1
27 August 2, 2001
If $f(x)=1+2(x-8)^{\frac{2}{3}}$ show that $f(0)=f(16)$but $f^{\backslash}(c)\ne 0$ for every number C in the interval $(0\,,16)$ Why doesn't this contradict Rolle's theorem?

2
45 10 May, 2009
Let $f(x)=6(2-x)^{4/3}+5$.

(a) Show that there is no $c\in(1\,,2)$ such that $f'(c)=0$.

(b) Does this contradict Rolle's Theorem for f on $[1\,,2]$? Explain.

Homework

1

37 June 6, 2010

Let $f(x) = x - \frac{3}{2}x^{\frac{2}{3}}$

(a) Find $f'(x)$

(b) Determine whether there exists a number $c \in (-8, 8)$ for which $f'(c) = 0$

(c) Which of the condition(s) of Rolle's Theorem are satisfied on $[-8, 8]$?
(justify your answer)